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A FINITE DIFFERENCE SCHEME FOR THE APPROXIMATION OF THE THIRD INITIAL BOUNDARY VALUE PARABOLIC PROBLEM*

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Abstract. We investigate the convergence of difference schemes that approximate the third initial boundary value problem for parabolic equations with time dependent coefficients. An abstract operator method is developed to analyze this equation. An estimate of the rate of the convergence in a special discrete $W_2^{1,1/2}$ Sobolev norm, compatible with the smoothness of the solution is obtained.

Key words. parabolic initial boundary value problem, oblique derivative boundary condition, finite differences, Sobolev spaces, convergence rate estimates

AMS subject classifications. 65M15

1. Introduction. For a class of finite difference schemes (FDSs) that approximate the parabolic boundary value problems (BVPs) with generalized solutions, convergence rate estimates compatible with the smoothness of the data

$$\|u - v\|_{W_p^{k,k/2}(Q_{h\tau})} \leqslant C \left(h^{s-k} + \tau^{\frac{s-k}{2}} \right) \|u\|_{W_p^{s,s/2}(Q)}, \qquad s \geqslant k,$$

are of particular interest, where $u = u(x)$ denotes the solution of the BVP defined on the space-time domain Q, v denotes the solution of the corresponding FDS defined on a finite difference mesh $Q_{h\tau}$, h and τ are the discretization parameters, $W_p^{k,\,k/2}(Q_{h\tau})$ is the Sobolev space of mesh functions, and C is a positive generic constant, independent of h, τ and u. A standard technique for establishing estimates of this types (see [9, 16, 20]) is based on the Bramble-Hilbert lemma [6, 8]. In the case of equations with variable coefficients, the constant C in the error bounds depends on the norms of the coefficients, as examined in [1, 2, 9, 16]. One-dimensional parabolic problems on the domain $Q = (0, 1) \times (0, T)$, are studied in [3, 4, 12], while 2D parabolic problems on the domain $Q = (0, 1)^2 \times (0, T)$ with variable coefficients (but not time-dependent) are considered in [4, 13]. A parabolic problem with time-dependent coefficients is investigated in [5, 17, 18, 19].

For the BVPs with an oblique derivative boundary condition, a loss of a half order in the convergence rate (usually $O(h^{3/2})$) is often observed. It is caused by the approximation of the boundary condition. Nevertheless, improvements were obtained in some cases, mainly for elliptic problems, as in [7, 11]. A type of parabolic problem in the two space dimensions with variable coefficients (but not time-dependent) is studied in [10].

In this paper, an error bound in the discrete $W_2^{1,1/2}$ norm is obtained under minimal smoothness assumptions on the data. It demonstrates second order convergence in the spatial discretisation parameter h and first order convergence in the temporal discretisation parameter τ for the FDS approximating an initial boundary value problem (IBVP) for a parabolic equation with time-dependent coefficients and an oblique derivative boundary condition.

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2. Formulation of the problem. As a model problem we consider the following third initial-boundary value problem for a parabolic equation in $Q = \Omega \times (0, T) = (0, 1)^2 \times (0, T)$ with variable coefficients:

$$
\begin{cases} \frac{\partial u}{\partial t} + Lu = f, & (x,t) = (x_1, x_2, t) \in Q, \\ lu = 0, & (x,t) \in \Gamma \times (0,T) = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.1}
$$

where

$$
Lu := -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right), \quad lu := \sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_j}\cos(\nu, x_i) + \alpha u
$$

and ν is the unit outward normal to Γ. We assume that the condition of strong ellipticity is satisfied:

$$
\begin{cases} a_{ij} = a_{ij}(x,t) = a_{ji}, & \alpha = \alpha(x), \\ c_0 \sum_{i=1}^{2} \xi_i^2 \leqslant \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \leqslant c_1 \sum_{i=1}^{2} \xi_i^2, & x \in \bar{\Omega}, \quad \forall \xi \in \mathbb{R}^2, \quad c_i = \text{const.} > 0, \\ \alpha_0 \leqslant \alpha(x) \leqslant \alpha_1, & \alpha_i = \text{const.} > 0. \end{cases}
$$

Let us define $\Gamma = \cup_{i=1}^{2} \cup_{k=0}^{1} \Gamma_{ik}$, where $\Gamma_{1k} = \{k\} \times [0,1]$, $\Gamma_{2k} = [0,1] \times \{k\}$, and $\Sigma_{ik} = \Gamma_{ik} \times [0,T]$. We also assume that the generalized solution of the problem (2.1) belongs to the Sobolev space $W_2^{3,3/2}(Q)$, while the data satisfy the following smoothness conditions

$$
\begin{cases} a_{ij} \in W_2^{2+\epsilon, 1+\epsilon/2}(Q), \\ \alpha \in W_2^{3/2}(\Gamma_{ik}), \\ \alpha \in C(\Gamma), \\ f \in W_2^{1,1/2}(Q), \\ u_0 \in W_2^2(\Omega). \end{cases}
$$

3. Finite difference approximation. Let $n, m \in \mathbb{N}$, $n \geqslant 2$, $m \geqslant 1$, $h = 1/n$, and $\tau = T/m$. We consider the uniform spatial mesh $\bar{\omega}$ with mesh size h on $\bar{\Omega}$ and the uniform temporal mesh $\bar{\omega}_\tau$ with mesh size τ on $[0,T]$. We also define

$$
\begin{array}{llll}
\omega = \bar{\omega} \cap \Omega, & \omega_\tau = \bar{\omega}_\tau \cap (0,T), & \omega_\tau^- = \bar{\omega}_\tau \cap [0,T), & \omega_\tau^+ = \bar{\omega}_\tau \cap (0,T], \\
\gamma = \bar{\omega} \cap \Gamma, & & \bar{\gamma}_{ik} = \bar{\omega} \cap \Gamma_{ik}, & \\
\gamma_{ik} = \{x \in \bar{\gamma}_{ik} : 0 < x_{3-i} < 1\}, & & \gamma_{ik}^- = \{x \in \bar{\gamma}_{ik} : 0 \leqslant x_{3-i} < 1\}, & \\
\gamma_{ik}^+ = \{x \in \bar{\gamma}_{ik} : 0 < x_{3-i} \leqslant 1\}, & & \gamma_{ik}^\star = \bar{\gamma}_{ik} \setminus \gamma_{ik}, & \\
\gamma^\star = \gamma \setminus \{\cup_{i,k}\gamma_{ik}\}, & & \sigma_{ik} = \gamma_{ik} \times \omega_\tau^+, & \bar{\sigma}_{ik} = \bar{\gamma}_{ik} \times \omega_\tau^+,
\end{array}
$$

where $i = 1, 2$, $k = 0, 1$, and $\bar{Q}_{h\tau} = \bar{\omega} \times \bar{\omega}_\tau$.

The finite difference operators are defined in the usual manner [15]:

$$
v_{x_i} = (v^{+i} - v)/h, \quad v_{\bar{x}_i} = (v - v^{-i})/h, \quad v_t = (\hat{v} - v)/\tau, \quad v_{\bar{t}} = (v - \check{v})/\tau,
$$

where $v^{\pm i}(x,t) = v(x \pm he_i, t)$, e_i is the unit vector of the axis x_i, $\hat{v}(x,t) = v(x, t+\tau)$, and $\check{v}(x,t) = v(x, t-\tau)$.

We also define the Steklov smoothing operators with step sizes h and τ [16]:

$$\begin{aligned} T_i^+ f(x,t) &= \int_0^1 f(x+hx'e_i,t)dx' = T_i^- f(x+he_i,t) = T_i f(x+\tfrac{h}{2}e_i,t),\\ T_i^{2\pm} f(x,t) &= 2\int_0^1 (1-x')f(x\pm hx'e_i)dx', \quad i=1,2,\\ T_t^+ f(x,t) &= \int_0^1 f(x,t+\tau t')dt' = T_t^- f(x,t+\tau) = T_i f(x,t+\tfrac{\tau}{2}). \end{aligned}$$

These operators commute and transform derivatives into divided differences. For example,

$$T_i^+\left(\frac{\partial u}{\partial x_i}\right) = u_{x_i}, \quad T_i^-\left(\frac{\partial u}{\partial x_i}\right) = u_{\bar{x}_i}, \quad T_i^2\left(\frac{\partial^2 u}{\partial x_i^2}\right) = u_{\bar{x}_i x_i}, \qquad i=1,2,$$

$$T_t^+\left(\frac{\partial u}{\partial t}\right) = u_t, \qquad T_t^-\left(\frac{\partial u}{\partial t}\right) = u_{\bar{t}}.$$

We approximate the IBVP (2.1) by the following implicit FDS:

$$\begin{aligned} &v_{\bar{t}} + L_h v = \tilde{f}, && x\in\bar{\omega}, \quad t\in\omega_\tau^+,\\ &v(x,0) = u_0(x), && x\in\bar{\omega}, \end{aligned} \tag{3.1}$$

where

$$L_h v = \begin{cases} -\frac{1}{2}\sum\limits_{i,j=1}^{2}\left[\left(a_{ij}v_{x_j}\right)_{\bar{x}_i} + \left(a_{ij}v_{\bar{x}_j}\right)_{x_i}\right], & x\in\omega,\\ \frac{2}{h}\left(-\frac{a_{11}+a_{11}^{+1}}{2}v_{x_1} - a_{12}\frac{v_{x_2}+v_{\bar{x}_2}}{2} + \tilde{\alpha}v\right) - \left(a_{12}v_{\bar{x}_2}\right)_{x_1} & \\ \qquad -\left(a_{21}v_{x_1}\right)_{\bar{x}_2} - \frac{1}{2}\left(a_{22}v_{x_2}\right)_{\bar{x}_2} - \frac{1}{2}\left(a_{22}v_{\bar{x}_2}\right)_{x_2}, & x\in\gamma_{10},\\ \frac{2}{h}\left[-\frac{a_{11}+a_{11}^{+1}}{2}v_{x_1} - a_{12}v_{x_2} - a_{21}v_{x_1} - \frac{a_{22}+a_{22}^{+2}}{2}v_{x_2}\right. & \\ \qquad \left. +(\tilde{\alpha}_1+\tilde{\alpha}_2)v\right], & x=(0,0),\\ \frac{2}{h}\left[-\frac{a_{11}+a_{11}^{+1}}{2}v_{x_1} - a_{12}v_{\bar{x}_2} + a_{21}v_{x_1} + \frac{a_{22}+a_{22}^{-2}}{2}v_{\bar{x}_2}\right. & \\ \qquad \left. +(\tilde{\alpha}_1+\tilde{\alpha}_2)v\right] - 2\left(a_{12}v_{\bar{x}_2}\right)_{x_1} - 2\left(a_{21}v_{x_1}\right)_{\bar{x}_2}, & x=(0,1). \end{cases}$$

Analogously, at the other boundary nodes $x\in\gamma\setminus\bar{\gamma}_{10}$,

$$\tilde{f} = \begin{cases} T_1^2 T_2^2 T_t^- f, & x\in\omega,\\ T_i^{2\pm} T_{3-i}^2 T_t^- f, & x\in\gamma_{i,0.5\mp0.5},\\ T_1^{2\pm} T_2^{2\pm} T_t^- f, & x=(0.5\mp0.5, 0.5\mp0.5)\in\gamma^\star, \end{cases}$$

and

$$\begin{cases} \tilde{\alpha} = T_{3-i}^2\alpha, \quad x\in\gamma_{i0}\cup\gamma_{i1}, & i=1,2,\\ \tilde{\alpha}_i = T_i^{2\pm}\alpha, \quad x\in\gamma^\star, \quad x_i = 0.5\mp0.5, & i=1,2. \end{cases}$$

4. Error analysis. Let u be the solution of the IBVP (2.1), and let v denote the solution of the FDS (3.1). The error $z = u - v$ is defined on $\bar{Q}_{h\tau}$ and satisfies the following conditions:

$$\begin{aligned} & z_{\bar{t}} + L_h z = \psi, \quad x \in \bar{\omega}, \quad t \in \omega_\tau^+, \\ & z(x,0) = 0, \qquad x \in \bar{\omega}, \end{aligned} \tag{4.1}$$

where

$$\psi = \begin{cases} \xi_{\bar{t}} + \sum_{i,j=1}^{2} \eta_{ij,\bar{x}_i}, & x \in \omega, \\ \tilde{\xi}_{\bar{t}} + \dfrac{2}{h}\eta_{11} + \dfrac{2}{h}\eta_{12} + \tilde{\eta}_{21,\bar{x}_2} + \tilde{\eta}_{22,\bar{x}_2} + \dfrac{2}{h}\zeta, & x \in \gamma_{10}, \\ \tilde{\tilde{\xi}}_{\bar{t}} + \dfrac{2}{h}\tilde{\eta}_{11} + \dfrac{2}{h}\tilde{\eta}_{12} + \dfrac{2}{h}\tilde{\eta}_{21} + \dfrac{2}{h}\tilde{\eta}_{22} + \dfrac{2}{h}(\zeta_1 + \zeta_2), & x = (0,0). \end{cases}$$

Analogously, at the other boundary nodes $x \in \gamma \setminus \gamma_{10}^-$, we have

$$\begin{aligned} & \xi = u - T_1^2 T_2^2 u, && x \in \omega, \\ & \tilde{\xi} = u - T_i^2 T_{3-i}^{2\pm} u, && x \in \gamma_{3-i,0.5\mp0.5}, \\ & \tilde{\tilde{\xi}} = u - T_1^{2\pm} T_2^{2\pm} u, && x = (0.5 \mp 0.5, 0.5 \mp 0.5) \in \gamma^\star, \end{aligned}$$

$$\eta_{ij} = T_i^+ T_{3-i}^2 T_t^- \left(a_{ij} \frac{\partial u}{\partial x_j} \right) - \frac{1}{2} \left(a_{ij} u_{x_j} + a_{ij}^{+i} u_{\bar{x}_j}^{+i} \right), \; x \in \omega,$$

$$\tilde{\eta}_{ii} = T_i^+ T_{3-i}^{2\pm} T_t^- \left(a_{ii} \frac{\partial u}{\partial x_i} \right) - \frac{a_{ii} + a_{ii}^{+i}}{2} u_{x_i}, \qquad x \in \gamma_{3-i,0.5\mp0.5}^-,$$

$$\tilde{\eta}_{i,3-i} = \begin{cases} T_i^+ T_{3-i}^{2+} T_t^- \left(a_{i,3-i} \dfrac{\partial u}{\partial x_{3-i}} \right) - a_{i,3-i} u_{x_{3-i}}, & x \in \gamma_{3-i,0}^-, \\ T_i^+ T_{3-i}^{2-} T_t^- \left(a_{i,3-i} \dfrac{\partial u}{\partial x_{3-i}} \right) - a_{i,3-i}^{+i} u_{\bar{x}_{3-i}}^{+i}, & x \in \gamma_{3-i,1}^-, \end{cases}$$

and

$$\begin{cases} \zeta = (T_i^2 \alpha) u - T_i^2 T_t^- (\alpha u), & x \in \gamma_{3-i,0} \cup \gamma_{3-i,1}, \\ \zeta_i = (T_i^{2\pm} \alpha) u - T_i^{2\pm} T_t^- (\alpha u), & x \in \gamma^\star, \quad x_i = 0.5 \mp 0.5. \end{cases}$$

We define the following discrete inner products:

$$\begin{cases} [v,w] = h^2 \sum_{x\in\omega} v(x)w(x) + \frac{h^2}{2} \sum_{x\in\gamma\setminus\gamma^\star} v(x)w(x) + \frac{h^2}{4} \sum_{x\in\gamma^\star} v(x)w(x), \\ [v,w)_i = h^2 \sum_{x\in\omega\cup\gamma_{i0}} v(x)w(x) + \frac{h^2}{2} \sum_{x\in\gamma_{3-i,0}^-\cup\gamma_{3-i,1}^-} v(x)w(x), \\ (v,w]_i = h^2 \sum_{x\in\omega\cup\gamma_{i1}} v(x)w(x) + \frac{h^2}{2} \sum_{x\in\gamma_{3-i,0}^+\cup\gamma_{3-i,1}^+} v(x)w(x), \\ [v,w) = h^2 \sum_{x\in\omega\cup\gamma_{10}^-\cup\gamma_{20}^-} v(x)w(x), \end{cases}$$

$$\begin{cases} (v,w] = h^2 \sum_{x\in\omega\cup\gamma_{11}^+\cup\gamma_{21}^+} v(x)w(x), \\ [v,w]_{\bar\gamma_{ik}} = h\sum_{x\in\gamma_{ik}} v(x)w(x) + \frac{h}{2}\sum_{x\in\gamma_{ik}^\star} v(x)w(x), \\ [v,w)_{\gamma_{ik}^-} = h\sum_{x\in\gamma_{ik}^-} v(x)w(x), \end{cases}$$

and the following norms:

$$\begin{cases} |[v]|^2 = [v,v], \quad |[v\|_i^2 = [v,v)_i, \quad \|v]|_i^2 = (v,v]_i, \quad |[v\|^2 = [v,v), \quad \|v]|^2 = (v,v], \\ |[v]|^2_{W_2^1(\bar\omega)} = |[v]|^2 + |[v_{x_1}\|_1^2 + |[v_{x_2}\|_2^2, \\ |[v]|^2_{\bar\gamma_{ik}} = [v,v]_{\bar\gamma_{ik}}, \quad |[v\|^2_{\gamma_{ik}^-} = [v,v)_{\gamma_{ik}^-}, \quad \|v\|^2_{\gamma_{ik}} = h\sum_{x\in\gamma_{ik}} v^2(x), \\ |v|^2_{W_2^{1/2}(\gamma_{ik}^-)} = h^2\sum_{x,x'\in\gamma_{ik}^-, x'\neq x}\left[\frac{v(x)-v(x')}{x_{3-i}-x'_{3-i}}\right]^2, \\ |[v\|^2_{W_2^{1/2}(\gamma_{ik}^-)} = |v|^2_{W_2^{1/2}(\gamma_{ik}^-)} + |[v\|^2_{\gamma_{ik}^-}, \\ |[v\|^2_{\ddot W_2^{1/2}(\gamma_{ik}^-)} = |v|^2_{W_2^{1/2}(\gamma_{ik}^-)} + h\sum_{x\in\gamma_{ik}^-}\left(\frac{1}{x_{3-i}+h/2} + \frac{1}{1-x_{3-i}-h/2}\right)v^2(x), \\ \|v\|_\tau^2 = \tau\sum_{t\in\omega_\tau^+} v^2(t), \quad |[v\|^2_{i,h\tau} = \tau\sum_{t\in\omega_\tau^+} |[v(\cdot,t)\|_i^2, \\ \|v\|^2_{\sigma_{ik}} = \tau\sum_{t\in\omega_\tau^+}\|v(\cdot,t)\|^2_{\gamma_{ik}}, \quad |[v]|^2_{\bar\sigma_{ik}} = \tau\sum_{t\in\omega_\tau^+}|[v(\cdot,t)]|^2_{\bar\gamma_{ik}}, \\ |[v\|^2_{L_2(\omega_\tau^+,\ddot W_2^{1/2}(\gamma_{ik}^-))} = \tau\sum_{t\in\omega_\tau^+}|[v(\cdot,t)\|^2_{\ddot W_2^{1/2}(\gamma_{ik}^-)}, \\ |v|^2_{W_2^{1/2}(\bar\omega_\tau,L_2(\bar\omega))} = \tau^2\sum_{t,t'\in\bar\omega_\tau,t'\neq t}\frac{|[v(\cdot,t)-v(\cdot,t')]|^2}{(t-t')^2}, \\ |[v]|^2_{\ddot W_2^{1/2}(\bar\omega_\tau,L_2(\bar\omega))} = |v|^2_{W_2^{1/2}(\bar\omega_\tau,L_2(\bar\omega))} + \tau\sum_{t\in\bar\omega_\tau}\left(\frac{1}{t+\tau/2}+\frac{1}{T-t+\tau/2}\right)|[v(\cdot,t)]|^2, \\ |[v]|^2_{L_2(\omega_\tau^+,W_2^1(\bar\omega))} = \tau\sum_{t\in\omega_\tau^+}|[v(\cdot,t)]|^2_{W_2^1(\bar\omega)}, \\ |[v]|^2_{W_2^{1,1/2}(Q_{h\tau})} = |[v]|^2_{L_2(\omega_\tau^+,W_2^1(\bar\omega))} + |v|^2_{W_2^{1/2}(\bar\omega_\tau,L_2(\bar\omega))}. \end{cases}$$

We shall prove a suitable a priori estimate for the FDS (4.1) which will be used to estimate its convergence rate.

The following lemma from [10] holds.

LEMMA 4.1. *Let* $w\in W_2^r(\Gamma_{ik})$, $0<r\leqslant 0.5$. *Then*

$$|T_{3-i}^+ w|_{W_2^{1/2}(\gamma_{ik}^-)} \leqslant C(r)h^{r-1/2}|w|_{W_2^r(\Gamma_{ik})}.$$

Let us rearrange the terms in the truncation error ψ in the following manner:

$$\tilde\eta_{ij} = \eta_{ij} + \eta'_{ij}, \qquad \tilde\xi = \xi + \xi', \qquad \tilde{\tilde\xi} = \xi + \xi^\star,$$

where

$$\eta'_{ii} = \pm\frac{h}{3}T_i^+T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(a_{ii}\frac{\partial u}{\partial x_i}\right)\right), \qquad x \in \gamma_{3-i,0.5\mp 0.5},$$

$$\eta'_{i,3-i} = \pm\frac{h}{3}T_i^+T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(a_{i,3-i}\frac{\partial u}{\partial x_{3-i}}\right)\right) \mp \frac{h}{2}T_i^+T_t^-\left(a_{i,3-i}\frac{\partial^2 u}{\partial x_{3-i}^2}\right)$$

$$+\frac{h}{2}T_i^+T_t^-\left(\frac{\partial}{\partial x_i}\left(a_{i,3-i}\frac{\partial u}{\partial x_{3-i}}\right)\right), \quad x \in \gamma_{3-i,\,0.5\mp 0.5},$$

$$\xi' = \mp\frac{h}{3}T_{3-i}^2\left(\frac{\partial u}{\partial x_i}\right), \qquad x \in \gamma_{i,\,0.5\mp 0.5},$$

$$\xi^\star = \mp\frac{h}{3}T_2^{2\pm}\left(\frac{\partial u}{\partial x_1}\right) \mp \frac{h}{3}T_1^{2\pm}\left(\frac{\partial u}{\partial x_2}\right), \qquad x = (0.5\mp 0.5,\ 0.5\mp 0.5) \in \gamma^\star.$$

Using the boundary condition in (2.1), we further obtain

$$\xi'_{\bar{t}} = \lambda_{i,\,\bar{x}_{3-i}} + \mu_i + \nu_i, \qquad x \in \gamma_{i,\,0.5\mp 0.5},$$

where

$$\lambda_i = \pm\frac{h}{3}T_{3-i}^+T_t^-\left(\frac{a_{i,3-i}}{a_{ii}}\frac{\partial u}{\partial t}\right),$$

$$\mu_i = \mp\frac{h}{3}T_{3-i}^2T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(\frac{a_{i,3-i}}{a_{ii}}\right)\frac{\partial u}{\partial t}\right),$$

$$\nu_i = -\frac{h}{3}T_{3-i}^2T_t^-\left(\frac{\alpha}{a_{ii}}\frac{\partial u}{\partial t}\right).$$

Similarly, for $x = (0,0)$, we obtain

$$\xi^\star_{\bar{t}} = \frac{2}{h}\lambda_1 - \frac{2}{h}\lambda_1^\star + \mu_1 + \nu_1 + \frac{2}{h}\lambda_2 - \frac{2}{h}\lambda_2^\star + \mu_2 + \nu_2,$$

where λ_1 and λ_2 are the same as before and

$$\lambda_i^\star = \pm\frac{h}{3}T_t^-\left(\frac{a_{i,3-i}}{a_{ii}}\frac{\partial u}{\partial t}\right),$$

$$\mu_i = -\frac{h}{3}T_{3-i}^{2+}T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(\frac{a_{i,3-i}}{a_{ii}}\right)\frac{\partial u}{\partial t}\right),$$

$$\nu_i = -\frac{h}{3}T_{3-i}^{2+}T_t^-\left(\frac{\alpha}{a_{ii}}\frac{\partial u}{\partial t}\right),$$

with an analogous representation at the other nodes of $\gamma^\star$.

The following a priori estimate from [10] is valid.

THEOREM 4.2. *The finite difference scheme* (4.1) *is stable in the sense of the a priori estimate*

$$
\begin{aligned}
|[z]|_{W_2^{1,\,1/2}(Q_{h\tau})} \leqslant C\Big\{ & |[\xi]|_{\ddot{W}_2^{1/2}(\bar\omega_\tau, L_2(\bar\omega))} + \sum_{i,j=1}^{2} |[\eta_{ij}\|_{i,h\tau} + \sum_{k=0}^{1}\sum_{i=1}^{2} \|\zeta\|_{\sigma_{ik}} \\
& + h\sum_{k=0}^{1}\sum_{i,j=1}^{2} |[\eta'_{ij}\|_{L_2(\omega_\tau^+,\, \ddot{W}_2^{1/2}(\gamma_{3-i,k}^-))} \\
& + h\sum_{k=0}^{1}\sum_{i=1}^{2} |[\lambda_i\|_{L_2(\omega_\tau^+,\, \ddot{W}_2^{1/2}(\gamma_{ik}^-))} \\
& + h\sum_{k=0}^{1}\sum_{i=1}^{2} (|[\mu_i]|_{\bar\sigma_{ik}} + |[\nu_i]|_{\bar\sigma_{ik}}) \\
& + h\sqrt{\log\tfrac{1}{h}}\sum_{i=1}^{2}\sum_{x\in\gamma^\star} (\|\zeta_i(x,\cdot)\|_\tau + \|\lambda_i^\star(x,\cdot)\|_\tau) \Big\}.
\end{aligned}
\tag{4.2}
$$

In accordance with Theorem 4.2, the problem of deriving a convergence rate estimate for the FDS (3.1) is reduced to estimating the right-hand side terms in the inequality (4.2).

Let us assume that $\tau \asymp h^2$, i.e., $c_2 h^2 \leqslant \tau \leqslant c_3 h^2$ for some positive constants c_2 and c_3.

The following estimates for the terms ζ_i, ζ and ν_i are proved in [10]:

$$
\begin{cases}
\|\zeta_i\|_\tau \leqslant Ch\|\alpha\|_{W_2^{3/2}(\Gamma_{3-i,k})}\|u\|_{W_2^{3,3/2}(Q)}, \\
\|\zeta\|_{\sigma_{3-i,k}} \leqslant Ch^2\|\alpha\|_{W_2^{3/2}(\Gamma_{3-i,k})}\|u\|_{W_2^{3,3/2}(Q)}, \\
|[\nu_i]|_{\bar\sigma_{ik}} \leqslant Ch\|u\|_{W_2^{3,3/2}(Q)}.
\end{cases}
\tag{4.3}
$$

Let us estimate the term η_{ij} at the internal mesh nodes. We decompose

$$\eta_{ij} = \eta_{ij,1} + \eta_{ij,2} + \eta_{ij,3},$$

where

$$
\begin{aligned}
\eta_{ij,1} &= T_i^+ T_{3-i}^2 T_t^- \left(a_{ij}\frac{\partial u}{\partial x_j}\right) - \left(T_i^+ T_{3-i}^2 T_t^- a_{ij}\right)\left(T_i^+ T_{3-i}^2 T_t^- \frac{\partial u}{\partial x_j}\right), \\
\eta_{ij,2} &= \left[T_i^+ T_{3-i}^2 T_t^- a_{ij} - 0,5\left(a_{ij} + a_{ij}^{+i}\right)\right]\left(T_i^+ T_{3-i}^2 T_t^- \frac{\partial u}{\partial x_j}\right), \\
\eta_{ij,3} &= -0,5\left(a_{ij} + a_{ij}^{+i}\right)\left\{T_i^+ T_{3-i}^2 T_t^- \frac{\partial u}{\partial x_j} - u_{x_j}\right\}.
\end{aligned}
$$

The term $\eta_{ij,1}$ is a bounded bilinear functional of the argument

$$(a_{ij}, u) \in W_4^{1,\,1/2}(e) \times W_4^{2,\,1}(e)$$

where

$$e = e(x,t) = \{(x_1', x_2', t') : x_i < x_i' < x_i + h, |x_{3-i}' - x_{3-i}| < h, t' \in (t-\tau, t)\}.$$

Further, $\eta_{ij,1} = 0$ whenever a_{ij} is a constant or u is a polynomial of degree one in x_1 or x_2 or a constant. By applying the Bramble-Hilbert lemma [6], we get:

$$|\eta_{ij,1}(x,t)| \leqslant C|a_{ij}|_{W_4^{1,\,1/2}(e)}|u|_{W_4^{2,\,1}(e)}\,. \tag{4.4}$$

The term $\eta_{ij,2}$ is a bounded bilinear functional of the argument

$$(a_{ij}, u) \in W_q^{2,\,1}(e) \times W_{2q/(q-2)}^{1,\,1/2}(e), \quad q = 2 + \varepsilon.$$

Further, $\eta_{ij,2} = 0$ whenever a_{ij} is a polynomial of degree one in x_1 or x_2 or constant or u is a constant. The application of the Bramble-Hilbert lemma allows one to get the following estimate:

$$|\eta_{ij,2}(x,t)| \leqslant C|a_{ij}|_{W_q^{2,\,1}(e)}|u|_{W_{2q/(q-2)}^{1,\,1/2}(e)}. \tag{4.5}$$

The term $\eta_{ij,3}$ is a bounded bilinear functional of the argument $(a_{ij}, u) \in \mathbb{C}(\overline{Q}) \times W_2^{3,\,3/2}(e)$. Further, $\eta_{ij,3} = 0$ whenever u is a polynomial of degree two in x_1 or x_2 and polynomial of arbitrary degree in t. By using the Bramble-Hilbert lemma, we get the following estimate:

$$|\eta_{ij,3}(x,t)| \leqslant C\|a_{ij}\|_{\mathbb{C}(\overline{Q})}|u|_{W_2^{3,\,3/2}(e)}\,. \tag{4.6}$$

From the estimates (4.4), (4.5), and (4.6), after summation and using the Sobolev imbeddings

$$\begin{aligned} W_2^{2+\varepsilon,1+\varepsilon/2} &\subset W_4^{1,1/2}, \\ W_2^{3,3/2} &\subset W_4^{2,1}, \\ W_2^{2+\varepsilon,\,1+\varepsilon/2} &\subset W_q^{2,1}, \\ W_2^{3,\,3/2} &\subset W_{2q/(q-2)}^{1,\,1/2}, \end{aligned}$$

for $q = 2 + \varepsilon$, $W_2^{2+\varepsilon,\,1+\varepsilon/2} \subset \mathbb{C}$, we have:

$$\left(\tau h^2 \sum_{t\in\omega_\tau^+}\sum_{x\in\omega\cup\gamma_{i0}} \eta_{ij}^2\right)^{1/2} \leqslant Ch^2\|a_{ij}\|_{W_2^{2+\varepsilon,\,1+\varepsilon/2}(Q)}\|u\|_{W_2^{3,\,3/2}(Q)}.$$

At the boundary nodes, η_{ii} can be decomposed in the following manner:

$$\eta_{ii} = \eta_{ii,1} + \eta_{ii,2} + \eta_{ii,3} + \eta_{ii,4}, \qquad x \in \gamma_{3-i,\,0.5\mp0.5}^-,$$

where

$$\eta_{ii,1} = T_i^+T_t^-\left(a_{ii}\,\frac{\partial u}{\partial x_i}\right) - \left(T_i^+T_t^-a_{ii}\right)\left(T_i^+T_t^-\,\frac{\partial u}{\partial x_i}\right),$$

$$\eta_{ii,2} = \left[\left(T_i^+T_t^-a_{ii}\right) - \frac{a_{ii}+a_{ii}^{+i}}{2}\right]\left(T_i^+T_t^-\,\frac{\partial u}{\partial x_i}\right),$$

$$\eta_{ii,3} = \frac{a_{ii}+a_{ii}^{+i}}{2}\left[\left(T_i^+T_t^-\,\frac{\partial u}{\partial x_i}\right) - \left(T_i^+\,\frac{\partial u}{\partial x_i}\right)\right],$$

$$\eta_{ii,4} = T_i^+T_{3-i}^{2\pm}T_t^-\left(a_{ii}\,\frac{\partial u}{\partial x_i}\right) - T_i^+T_t^-\left(a_{ii}\,\frac{\partial u}{\partial x_i}\right) \mp \frac{h}{3}\,T_i^+T_t^-\left(\frac{\partial}{\partial x_{3-i}}\left(a_{ii}\,\frac{\partial u}{\partial x_i}\right)\right).$$

The term $\eta_{11,1}$ is a bounded bilinear functional of the argument

$$(a_{11},u)\in W_q^{1,\,1/2}(e_{2k})\times W_{2q/(q-2)}^{3/2,\,3/4}(e_{2k}),\quad q>2,$$

where $e_{2k}=(x_1,x_1+h)\times\{k\}\times(t-\tau,t)$, $k=0,1$. Further, $\eta_{11,1}=0$ whenever a_{11} is a constant or u is a polynomial of degree one in x_1 or x_2 or a constant. By applying the Bramble-Hilbert lemma, we get the following estimate:

$$|\eta_{11,1}(x,t)|\leqslant C|a_{11}|_{W_q^{1,\,1/2}(e_{2k})}|u|_{W_{2q/(q-2)}^{3/2,\,3/4}(e_{2k})}\,.$$

After summation, using the trace theorem [5] and the following Sobolev imbeddings

$$W_2^{3/2,3/4}\subset W_q^{1,\,1/2},$$
$$W_2^{5/2,\,5/4}\subset W_{2q/(q-2)}^{3/2,\,3/4},$$

we obtain

$$\left(\tau h^2\sum_{t\in\omega_\tau^+}\sum_{x\in\gamma_{2k}^-}\eta_{11,1}^2\right)^{1/2}\leqslant Ch^2\|a_{11}\|_{W_2^{3/2,3/4}(\Sigma_{2k})}\|u\|_{W_2^{5/2,5/4}(\Sigma_{2k})}\tag{4.7}$$
$$\leqslant Ch^2\|a_{11}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}.$$

The term $\eta_{11,2}$ is a bounded bilinear functional of the argument

$$(a_{11},u)\in W_q^{3/2,\,3/4}(e_{2k})\times W_{2q/(q-2)}^{1,\,1/2}(e_{2k}),\ q>2.$$

Further, $\eta_{11,2}=0$ whenever a_{11} is a polynomial of degree one in x_1 or x_2 or a constant or u is a constant. The following estimate are obtained by means of the Bramble-Hilbert lemma:

$$|\eta_{11,2}(x,t)|\leqslant C|a_{11}|_{W_q^{3/2,\,3/4}(e_{2k})}|u|_{W_{2q/(q-2)}^{1,\,1/2}(e_{2k})}\,.$$

After summation, using the trace theorem and the imbedding $W_2^{5/2,\,5/4}\subset W_{2q/(q-2)}^{1,\,1/2}$ we obtain

$$\left(\tau h^2\sum_{t\in\omega_\tau^+}\sum_{x\in\gamma_{2k}^-}\eta_{11,2}^2\right)^{1/2}\leqslant Ch^2\|a_{11}\|_{W_2^{3/2,3/4}(\Sigma_{2k})}\|u\|_{W_2^{5/2,5/4}(\Sigma_{2k})}$$
$$\leqslant Ch^2\|a_{11}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}.$$

The term $\eta_{11,3}$ is a bounded bilinear functional of the argument

$$(a_{11},u)\in \mathrm{C}(\overline{Q})\times W_2^{5/2,\,5/4}(e_{2k}).$$

Further, $\eta_{11,3}=0$ whenever u is a polynomial of degree two in x_1 or x_2 and a polynomial of degree one in t. By applying the Bramble-Hilbert lemma, we get the following estimate:

$$|\eta_{11,3}(x,t)|\leqslant C\|a_{11}\|_{\mathbb{C}(\overline{Q})}|u|_{W_2^{5/2,\,5/4}(e_{2k})}\,.$$

After summation, using the trace theorem and the imbedding $W_2^{2+\varepsilon,1+\varepsilon/2} \subset \mathrm{C}$ we obtain

$$\left(\tau h^2 \sum_{t\in\omega_\tau^+} \sum_{x\in\gamma_{2k}^-} \eta_{11,3}^2\right)^{1/2} \leqslant Ch^2 \|a_{11}\|_{\mathbb{C}(\overline{Q})} \|u\|_{W_2^{5/2,5/4}(\Sigma_{2k})}$$
$$\leqslant Ch^2 \|a_{11}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)} \|u\|_{W_2^{3,3/2}(Q)}.$$

The term $\eta_{11,4}$ on γ_{20}^- is a bounded linear functional of $w = a_{11}\,\frac{\partial u}{\partial x_1} \in W_2^{2,\,1}(e)$, where $e = (x_1, x_1 + h) \times (0, h) \times (t - \tau, t)$. Further, $\eta_{11,4} = 0$ whenever $a_{11}\frac{\partial u}{\partial x_1}$ is a polynomial of degree one in x_1 or x_2 or a constant. By applying the Bramble-Hilbert lemma, we get the following estimate:

$$|\eta_{11,4}(x,t)| \leqslant C \left| a_{11}\frac{\partial u}{\partial x_1} \right|_{W_2^{2,\,1}(e)}.$$

After summation and using properties of multipliers in Sobolev spaces, we obtain

$$\left(\tau h^2 \sum_{t\in\omega_\tau^+} \sum_{x\in\gamma_{20}^-} \eta_{11,4}^2\right)^{1/2} \leqslant Ch^2 \left\| a_{11}\frac{\partial u}{\partial x_1} \right\|_{W_2^{2,\,1}(Q)}$$
$$\leqslant Ch^2 \|a_{11}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)} \|u\|_{W_2^{3,3/2}(Q)}.$$

Putting $e = (x_1, x_1 + h) \times (1 - h, h) \times (t - \tau, t)$, we have analogously

$$\left(\tau h^2 \sum_{t\in\omega_\tau^+} \sum_{x\in\gamma_{21}^-} \eta_{11,4}^2\right)^{1/2} \leqslant Ch^2 \left\| a_{11}\frac{\partial u}{\partial x_1} \right\|_{W_2^{2,\,1}(Q)} \tag{4.8}$$
$$\leqslant Ch^2 \|a_{11}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)} \|u\|_{W_2^{3,3/2}(Q)}.$$

From (4.7)–(4.8), we have

$$\left(\tau h^2 \sum_{t\in\omega_\tau^+} \sum_{x\in\gamma_{20}^-\cup\gamma_{21}^-} \eta_{11}^2\right)^{1/2} \leqslant Ch^2 \|a_{11}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)} \|u\|_{W_2^{3,3/2}(Q)}. \tag{4.9}$$

An analogous estimate holds for the term η_{22}:

$$\left(\tau h^2 \sum_{t\in\omega_\tau^+} \sum_{x\in\gamma_{10}^-\cup\gamma_{11}^-} \eta_{22}^2\right)^{1/2} \leqslant Ch^2 \|a_{22}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)} \|u\|_{W_2^{3,3/2}(Q)}. \tag{4.10}$$

Similarly, at the boundary nodes $\eta_{i,3-i}$ can be decomposed in the following manner:

$$\eta_{i,3-i} = \eta_{i,3-i,1} + \eta_{i,3-i,2} + \eta_{i,3-i,3} + \eta_{i,3-i,4}, \qquad x \in \gamma_{3-i,\,0.5\mp0.5}^-,$$

where

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$$\eta_{i,3-i,1} = T_i^+ T_{3-i}^{2\pm} T_t^- \left(a_{i,3-i} \frac{\partial u}{\partial x_{3-i}} \right) - T_i^+ T_t^- \left(a_{i,3-i} \frac{\partial u}{\partial x_{3-i}} \right)$$
$$\mp \frac{h}{3} T_i^+ T_t^- \left(\frac{\partial}{\partial x_{3-i}} \left(a_{i,3-i} \frac{\partial u}{\partial x_{3-i}} \right) \right),$$
$$\eta_{i,3-i,2} = T_i^+ T_t^- \left(a_{i,3-i} \frac{\partial u}{\partial x_{3-i}} \right) - T_t^- \left(a_{i,3-i} \frac{\partial u}{\partial x_{3-i}} \right)$$
$$- \frac{h}{2} T_i^+ T_t^- \left(\frac{\partial}{\partial x_i} \left(a_{i,3-i} \frac{\partial u}{\partial x_{3-i}} \right) \right),$$
$$\eta_{i,3-i,3} = a_{i,3-i} \left[T_t^- \left(\frac{\partial u}{\partial x_{3-i}} \right) - T_{3-i}^{\pm} \left(\frac{\partial u}{\partial x_{3-i}} \right) \pm \frac{h}{2} T_i^+ T_t^- \left(\frac{\partial^2 u}{\partial x_{3-i}^2} \right) \right],$$
$$\eta_{i,3-i,4} = \pm \frac{h}{2} \left[T_i^+ T_t^- \left(a_{i,3-i} \frac{\partial^2 u}{\partial x_{3-i}^2} \right) - a_{i,3-i} T_i^+ T_t^- \left(\frac{\partial^2 u}{\partial x_{3-i}^2} \right) \right].$$

To estimate the term η_{12}, we apply the same technique as for the term $\eta_{11,4}$:

$$\left(\tau h^2 \sum_{t \in \omega_\tau^+} \sum_{x \in \gamma_{2k}^-} \eta_{12,1}^2 \right)^{1/2} \leqslant Ch^2 \|a_{12}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)} \|u\|_{W_2^{3,3/2}(Q)} \tag{4.11}$$

and

$$\left(\tau h^2 \sum_{t \in \omega_\tau^+} \sum_{x \in \gamma_{2k}^-} \eta_{12,2}^2 \right)^{1/2} \leqslant Ch^2 \|a_{12}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)} \|u\|_{W_2^{3,3/2}(Q)}.$$

The term $\eta_{12,3}$ is a bounded bilinear functional of the argument

$$(a_{12}, u) \in \mathrm{C}(\overline{Q}) \times W_2^{3,\,3/2}(e).$$

Further, $\eta_{12,3} = 0$ whenever u is a polynomial of degree two in x_1 or x_2 or a polynomial of degree one in t. By applying the Bramble-Hilbert lemma, we get the following estimate:

$$|\eta_{12,3}(x,t)| \leqslant C \|a_{12}\|_{\mathbb{C}(\overline{Q})} |u|_{W_2^{3,\,3/2}(e)}\,.$$

After summation and using the imbedding $W_2^{2+\varepsilon,\,1+\varepsilon/2} \subset \mathrm{C}(\overline{Q})$, we obtain

$$\left(\tau h^2 \sum_{t \in \omega_\tau^+} \sum_{x \in \gamma_{2k}^-} \eta_{12,3}^2 \right)^{1/2} \leqslant Ch^2 \|a_{12}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)} \|u\|_{W_2^{3,3/2}(Q)}.$$

The term $\eta_{12,4}$ can be estimated directly. Let us set $x = (x_1, 0) \in \gamma_{20}^-$. We have the representation

$$\eta_{12,4}(x_1,0,t) = \frac{h}{2} \frac{2}{h\tau} \int\limits_{x_1}^{x_1+h} \left(1 - \frac{x_1'}{h} \right) \int\limits_{t-\tau}^{t} \int\limits_{x_1}^{x_1'} \frac{\partial a_{12}}{\partial x_1}(x_1'', 0, t') \frac{\partial^2 u}{\partial x_2^2}(x_1', 0, t')\, dx_1'' dt' dx_1'.$$

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Hence, for $q=3$,

$$\left(\tau h^2 \sum_{t\in\omega_\tau^+}\sum_{x\in\gamma_{2k}^-}\eta_{12,4}^2\right)^{1/2} \leqslant Ch^2\|a_{12}\|_{W_q^{1,1/2}(\Sigma_{2k})}\|u\|_{W_{2q/(q-2)}^{1,1/2}(\Sigma_{2k})} \leqslant Ch^2\|a_{12}\|_{W_q^{3/2,3/4}(\Sigma_{2k})}\|u\|_{W_2^{5/2,5/4}(\Sigma_{2k})} \leqslant Ch^2\|a_{12}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}. \tag{4.12}$$

From (4.11)–(4.12) we have

$$\left(\tau h^2 \sum_{t\in\omega_\tau^+}\sum_{x\in\gamma_{20}^-\cup\gamma_{21}^-}\eta_{12}^2\right)^{1/2} \leqslant Ch^2\|a_{12}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}. \tag{4.13}$$

An analogous estimate holds for the term η_{21}:

$$\left(\tau h^2 \sum_{t\in\omega_\tau^+}\sum_{x\in\gamma_{10}^-\cup\gamma_{11}^-}\eta_{21}^2\right)^{1/2} \leqslant Ch^2\|a_{21}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}. \tag{4.14}$$

In such a way, from (4.9), (4.10), (4.13), and (4.14) one obtains

$$|[\eta_{ij}\|_{i,h\tau} \leqslant Ch^2\,\|a_{ij}\|_{W_2^{2+\varepsilon,\,1+\varepsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}. \tag{4.15}$$

Let us now estimate $|[\eta'_{11}\|_{L_2(\omega_\tau^+,\,\ddot{W}_2^{1/2}(\gamma_{20}^-))}$. By using Lemma 4.1, we immediately obtain

$$|\eta'_{11}|_{W_2^{1/2}(\gamma_{20}^-)} \leqslant Ch\left|T_t^-\left(\frac{\partial}{\partial x_2}\left(a_{11}\frac{\partial u}{\partial x_1}\right)\right)\right|_{W_2^{1/2}(\Gamma_{20})} \leqslant Ch\left\|T_t^-\left(\frac{\partial}{\partial x_2}\left(a_{11}\frac{\partial u}{\partial x_1}\right)\right)\right\|_{W_2^1(\Omega)}.$$

Using the inequality [14]

$$\|F\|_{L_2(0,\varepsilon)} \leqslant C\varepsilon^{1/2}\,\log\frac{1}{\varepsilon}\,\|F\|_{W_2^{1/2}(0,1)},$$

we obtain

$$h\sum_{x\in\gamma_{20}^-}\left(\frac{1}{x_1+h/2}+\frac{1}{1-x_1-h/2}\right)(\eta'_{11})^2 \leqslant Ch^2\log\frac{1}{h}\left\|T_t^-\left(\frac{\partial}{\partial x_2}\left(a_{11}\frac{\partial u}{\partial x_1}\right)\right)\right\|^2_{W_2^{1/2}(\Gamma_{20}^-)} \leqslant Ch^2\log\frac{1}{h}\left\|T_t^-\left(\frac{\partial}{\partial x_2}\left(a_{11}\frac{\partial u}{\partial x_1}\right)\right)\right\|^2_{W_2^1(\Omega)},$$

and

$$h \sum_{x\in\gamma_{20}^-} \left(\frac{1}{x_1+h/2}+\frac{1}{1-x_1-h/2}\right)(\eta_{11}')^2 \leqslant Ch^2\log^3\frac{1}{h}\left\|T_t^-\left(\frac{\partial}{\partial x_2}\left(a_{11}\frac{\partial u}{\partial x_1}\right)\right)\right\|^2_{W_2^1(\Omega)}.$$

From the obtained inequalities, summing over the mesh ω_τ^+, and using properties of multipliers in Sobolev spaces, we immediately obtain

$$\|\eta_{11}'\|_{L_2(\omega_\tau^+, W_2^{1/2}(\gamma_{20}^-))} \leqslant Ch\left(\log\tfrac{1}{h}\right)^{3/2}\|a_{11}\|_{W_2^{2+\epsilon,1+\epsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}.$$

Analogous estimates holds for the other η_{ij}, so we have

$$\|\eta_{ii}'\|_{L_2(\omega_\tau, \ddot W_2^{1/2}(\gamma_{3-i,k}^-))} \leqslant Ch\left(\log\tfrac{1}{h}\right)^{3/2}\|a_{ii}\|_{W_2^{2+\varepsilon,\,1+\varepsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)} \tag{4.16}$$

and

$$\|\eta_{i,3-i}'\|_{L_2(\omega_\tau, \ddot W_2^{1/2}(\gamma_{3-i,k}^-))} \leqslant Ch\left(\log\tfrac{1}{h}\right)^{3/2}\|a_{i,3-i}\|_{W_2^{2+\varepsilon,\,1+\varepsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}. \tag{4.17}$$

By using an analogous technique as in [10], we have the following estimates for the terms ξ, λ_i, $\lambda_i^\star$ and μ_i:

$$\begin{cases} [\xi]|_{\ddot W_2^{1/2}(\bar\omega_\tau,\,L_2(\bar\omega))} \leqslant Ch^2\sqrt{\log\tfrac{1}{h}}\,\|a_{ij}\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}, \\ \|\lambda_i\|_{L_2(\omega_\tau,\ddot W_2^{1/2}(\gamma_{ik}^-))} \leqslant Ch\left(\log\tfrac{1}{h}\right)^{3/2}\Big(\|a_{ii}\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q)} \\ \qquad + \|a_{i,3-i}\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q)}\Big)\|u\|_{W_2^{3,3/2}(Q)}, \\ \|\lambda_i^\star\|_\tau \leqslant Ch\|a_{i,3-i}\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q)}\|u\|_{W_2^{3,3/2}(Q)}, \\ |[\mu_i]|_{\bar\sigma_{ik}} \leqslant Ch\Big(\|a_{ii}\|_{W_2^{2+\varepsilon,1+\varepsilon/2}(Q)} \\ \qquad + \|a_{i,3-i}\|_{W_2^{2+\varepsilon,\,1+\varepsilon/2}(Q)}\Big)\|u\|_{W_2^{3,3/2}(Q)}. \end{cases} \tag{4.18}$$

From (4.2), (4.3), (4.15)–(4.18), we obtain the following result.

THEOREM 4.3. *Let the assumptions from Section 2 hold and let $\tau \asymp h^2$. Then the FDS (3.1) converges in the norm $W_2^{1,1/2}(Q_{h\tau})$ and the following convergence rate estimate holds:*

$$|[z]|_{W_2^{1,1/2}(Q_{h\tau})} \leqslant Ch^2\left(\log\tfrac{1}{h}\right)^{\frac{3}{2}}\left(1+\max_{i,j}\|a_{ij}\|_{W_2^{2+\varepsilon,\,1+\varepsilon/2}(Q)}+\max_{i,k}\|\alpha\|_{W_2^{3/2}(\Gamma_{ik})}\right)\|u\|_{W_2^{3,3/2}(Q)}.$$

REMARK 4.4. Note that the obtained error bound is "almost" compatible with the smoothness of the data (up to additional logarithmic factors).

REMARK 4.5. Assuming that the generalized solution of the problem belongs to the Sobolev space $W_2^{s,s/2}(Q)$, $2.5 < s \leqslant 3$, an estimate of the convergence rate of the finite difference scheme for the parabolic problem with variable coefficients (but not time-dependent) is obtained; see [10].

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